

# Dispersion-dissipation condition for finite difference schemes

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## Abstract

In this short note, by analyzing the dispersion and dissipation of explicit finite difference scheme, a dispersion-dissipation condition is derived to determine the minimum dissipation required to damp the artificial high-wavenumber waves in the solution. The example application to our previous developed WENO-CU6-M2 scheme suggests that this condition can be used as an general guidance on optimizing the dissipation of a numerical scheme.

*Key words:* dispersion, dissipation, finite difference scheme, weighted-essentially non-oscillatory scheme.

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## 1 Introduction

The spectral analysis of finite difference scheme shows that the dispersion and dissipation introduced by the scheme determine the errors related to the propagation of a simple wave with given wavenumber. While the original spectral analysis is only for linear schemes, Pirozzoli [9] developed a numerical spectral analysis for general non-linear finite difference schemes, such as shock-

capturing scheme. Recently, Sun et al. [11] found that the dispersion and dissipation of a class of the explicit linear schemes can be controlled independently. Since the pioneer works of Tam and Webb [12] on computational aero-acoustics the spectral analysis has been applied widely to the design of high-order numerical schemes, especially those for the direct numerical simulation (DNS) of turbulent flow, for accurately resolving small scale in both amplitude and phase [15, 14, 10, 5, 7].

While it is generally accepted that the dispersion of a numerical scheme should be minimized according to some chosen criteria, there are no general guidelines on how the dissipation should be optimized [11]. For example, it is known that the weighted essentially non-oscillatory (WENO) schemes using upwind schemes as the underlying optimal linear scheme are too dissipative for DNS. However, the minimum dissipation produced by the central scheme may not be sufficient to suppress numerical oscillations [6]. Pirozzoli [8] pointed out that a certain amount of dissipation is desirable to damp as much as possible the high-wavenumber waves with incorrect propagation speed. The difficulty is to determine how much dissipation a numerical scheme should provided to damp the artificial high-wavenumber waves, but not the physical waves to be resolved.

The objective of this short note is to show that the minimum sufficient dissipation of a numerical scheme is actually determined by its dispersion with a dispersion-dissipation condition. This condition is then applied to alleviate the difficult of artificial waves produced by our previous developed WENO-CU6-M2 scheme [2]. Two straightforward modifications are proposed together with the numerical example on the two-blast wave interaction problem.

## 2 Dispersion-dissipation condition

For completeness, we begin from the spectral analysis of the one-dimensional linear advection equation [13]

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad -\infty < x < +\infty, \quad u(x, 0) = e^{iw x}, \quad (1)$$

with the initial condition of a monochromatic sinusoidal with wavenumber  $w$  and unit amplitude where  $c$  is the advection speed. Eq. (1) is discretized in the spatial domain such that  $x_j = j\Delta x$ ,  $j = 0, \dots, N$ , where  $\Delta x$  is spacing and  $u_j = u(x_j)$  is the node value, and the semi-discretized form by the method of lines yields a system of ordinary differential equations

$$\frac{du_j}{dt} = -u'_j, \quad j = 0, \dots, N, \quad u_j(0) = e^{ij\phi}, \quad (2)$$

where  $\phi = w\Delta x$  is the reduced wavenumber, and  $u'_j \approx \left. \frac{\partial u}{\partial x} \right|_{x=x_j}$  is a finite difference approximation. If an explicit linear scheme is applied

$$u'_j = \frac{1}{\Delta x} \sum_{l=-q}^r a_l u_{j+l}, \quad (3)$$

the analytical solution of Eq. (2) is

$$u_j(t) = e^{-i\text{Im}(\Phi(\phi))c \frac{t}{\Delta x}} e^{iw(j\Delta x - \frac{\text{Re}(\Phi(\phi))}{\phi} ct)}, \quad (4)$$

where

$$\Phi(\phi) = -i \sum_{l=-q}^r a_l e^{il\phi} = \sum_{l=-q}^r a_l [\sin(l\phi) - i \cos(l\phi)], \quad (5)$$

is the modified wavenumber. If Eq. (4) is compared with the solution of the exact Eq. (1) at the  $x_j$

$$u_j(t) = e^{iw(j\Delta x - ct)}, \quad (6)$$

it is easy to find that the introduction of the finite difference approximation Eq. (3) leads two type of numerical errors. The first type of errors, called dissipation, modifies the amplitude of the wave with the factor  $e^{-\lambda t}$ , where  $\lambda = i\text{Im}(\Phi(\phi))c/\Delta x$  is the decay rate. In the classical theory of linear stability, if  $\alpha > 1$  or  $\lambda < 0$ , the dissipative error leads to the exponentially growth of the amplitude and the semi-discretization Eq. (2) is unstable. The second type errors, called dispersion, modifies the propagation speed of the wave to  $cc^*(\phi)$ , where  $c^*(\phi) = \text{Re}(\Phi(\phi))/\phi$  is the normalized phase velocity. Usually, for a complex wave packet with a wavenumber spectrum, the apparent velocity of a wave component with phase velocity  $c^*(\phi)$  is given by the group velocity  $\mathcal{V} = cd(\phi c^*(\phi))/d\phi = c\text{Re}(\Phi(\phi))'$ , Note that, while the wave is spreading away from the main wavefront with the speed  $|\Delta c| = |\mathcal{V} - c|$ , its magnitude is decreasing with the decay rate  $\lambda$ . Therefore, the distance  $\delta$  from the main wavefront in which the wave can survive is

$$\delta \sim \Delta x \frac{|\Delta c|}{i\text{Im}(\Phi(\phi))c}. \quad (7)$$

If the solution is free of artificial waves, i.e.,  $\delta \sim \Delta x$ , the condition must be satisfied is the dispersion-dissipation ratio

$$r = \frac{|\text{Re}(\Phi(\phi))' - 1|}{i\text{Im}(\Phi(\phi))} \sim 1, \quad (8)$$

if the dispersion error is not negligible. Since  $\delta$  decreases with increase of  $\lambda$ , Eq. (8) actually determines the minimum sufficient dissipation required to

damp the artificial waves. Specifically, for the explicit linear scheme Eq.(3), the above condition is

$$r = \frac{\left| \sum_{l=-q}^r a_l l \cos(l\phi) - 1 \right|}{\sum_{l=-q}^r a_l \cos(l\phi)} \sim 1. \quad (9)$$

Note that, for non-linear schemes, since the modified wavenumber Eq. (5) can be obtained approximated numerically by the approximated dispersion relation (ADR) [9], one can still define the dispersion dissipation condition by Eq. (8).

### 3 Application to WENO-CU6-M2 scheme

The WENO-CU6-M2 scheme [2] is an adaptive central-upwind weighted essentially non-oscillatory (WENO) scheme with 6th-order accuracy for smooth solutions. The scheme uses a 6-point stencil to achieve adaptively, by a non-linear weighting procedure, either the approximation of the optimal 6th-order central scheme in smooth regions of the solution, or a numerical stable 3rd-order approximation when there are discontinuities or shocks in the solution. The nonlinear weights are given by

$$\omega_r = \frac{\alpha_r}{\sum_{r=0}^3 \alpha_r}, \quad \alpha_r = d_r \left( C_q + \frac{\tau_6}{\beta_{3,r} + \epsilon \Delta x^2} \frac{\beta_{3,ave} + \chi \Delta x^2}{\beta_{3,r} + \chi \Delta x^2} \right)^q, \quad (10)$$

where  $d_r = \{0.05, 0.45, 0.45, 0.05\}$  are the optimal weights leading to the approximation of the 6th-order central scheme,  $q = 4$  is an integer parameter,  $C_q = 10^3$  is a positive constant parameter and  $\epsilon = 1/\chi = 10^{-8}$  is a small positive number. For the WENO methodology and the details of computing  $\beta_{3,r}$ ,  $\beta_{3,ave}$  and  $\tau_6$ , please refer to Refs. [4, 2]. As shown in Fig. 1a, the mod-

ified wavenumber of WENO-CU6-M2 is very close to that of the 6th-order central scheme until about  $\phi = \pi/2$ . Since the latter produces considerable dispersion beyond  $\phi = 1$  but without dissipation, as shown in Fig. 1b, the dispersion-dissipation ratio of WENO-CU6-M2 does not satisfy the condition Eq. (8) in the wavenumber range around  $\phi = 1.1$  to  $\pi/2$ , and has the maximum value about  $r = 100$  at  $\phi = 1.302$ . Therefore, WENO-CU6-M2 may produces artificial waves in such wavenumber range.

To illustrate the artificial waves in the solution produced by WENO-CU6-M2, the two-blast-wave interaction problem, which is taken from Woodward and Colella [1], is simulated on 400 grid points. The reference "exact" solution is a high-resolution numerical solution on 3200 grid points calculated by the WENO-CU6 scheme [3]. The numerical solution in Fig. 2 shows that, while WENO-CU6-M2 achieves very high resolvability for the density valley at about  $x = 0.75$  and peak at about  $x = 0.78$  (see Fig. 2a and b), small artificial waves can be observed in the close-up views of the velocity profile (see Fig. 2c and d). These waves have wavelengths about 4 to 6 grid points, which corresponds to the wavenumber range with large  $r$  as shown in Fig. 1b.

The difficulty of artificial waves can be alleviated either by increasing the dissipation (denoted as modification *A*), or by modifying both dispersion and dissipation (denoted as modification *B*), to satisfy the dispersion-dissipation relation of Eq. (8). For modification *A*, other than the 6th-order central scheme, the 5th-order linear scheme with the constraint  $r = 10$  at  $\phi = 1.302$  according to Eq. (9) is chosen as the optimal scheme. The resulting the optimal weights are  $d_r = \{0.065, 0.495, 0.405, 0.035\}$ . As shown in Fig. 1a and b, such modification dos not change the dispersion, but achieves  $r < 10$  in the entire wavenumber range. The numerical solution of the two-blast-wave interaction problem,

as shown in Fig. 2a and b, suggests no notable differences from those obtained by the original scheme. However, the artificial waves in the velocity profile are eliminated (see Fig. 2c and d). For modification  $B$ , a 4th-order scheme, whose dispersion is optimized by minimizing the integral error  $E = \int_0^\pi e^{6(\pi-\phi)} |c^*|^2 d\phi$  with the constraint  $c^* \leq 0.015$  for all  $\phi < \phi_c$ , where  $\phi_c = 1.5$  is a critical wavenumber, and whose dissipation is separately constrained by  $r = 10$  at  $\phi = \pi/2$  [11], is chosen as the underlying optimal scheme. The resulting optimal weights are  $d_r = \{0.0904546, 0.4440908, 0.3922727, 0.0731819\}$ . As shown in Fig. 1a, the modified scheme achieves both considerable improvement on dispersion, and  $r < 10$  in the entire wavenumber range. Again, as shown in Fig. 2 the numerical solution obtained by the modified scheme suggests no notable differences from those by the original scheme, and the artificial waves in the velocity profile are eliminated.

## 4 Concluding remarks

In this short note, we have derived the dispersion-dissipation condition which determines the minimum dissipation that a numerical sufficient finite difference scheme required to damp the artificial high-wavenumber waves in the solution. With the help of approximated dispersion relation, the condition can also be defined for non-linear schemes. The example application to WENO-CU6-M2 scheme suggests that this condition can be used as a general guidance on optimizing the dissipation of a numerical scheme.

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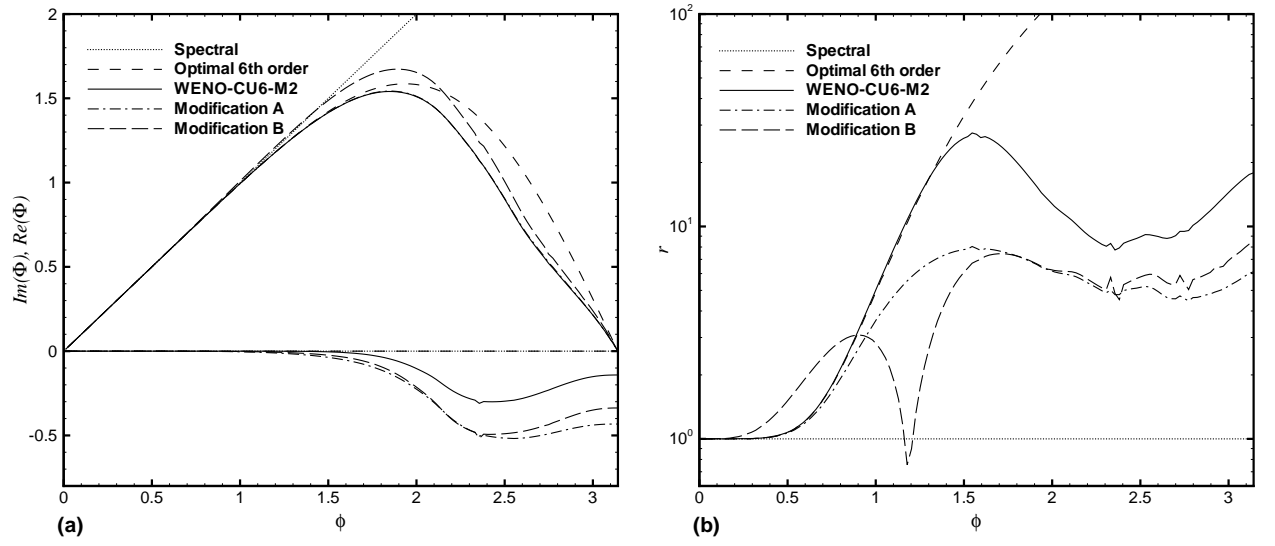


Figure 1. Approximated dispersion relation of various numerical schemes: (a) modified wavenumber and (b) dispersion-dissipation ratio computed with  $r = \frac{\text{Re}(\Phi(\phi))' + \varepsilon}{i\text{Im}(\Phi(\phi)) + \varepsilon}$ , where  $\varepsilon = 10^{-3}$ .

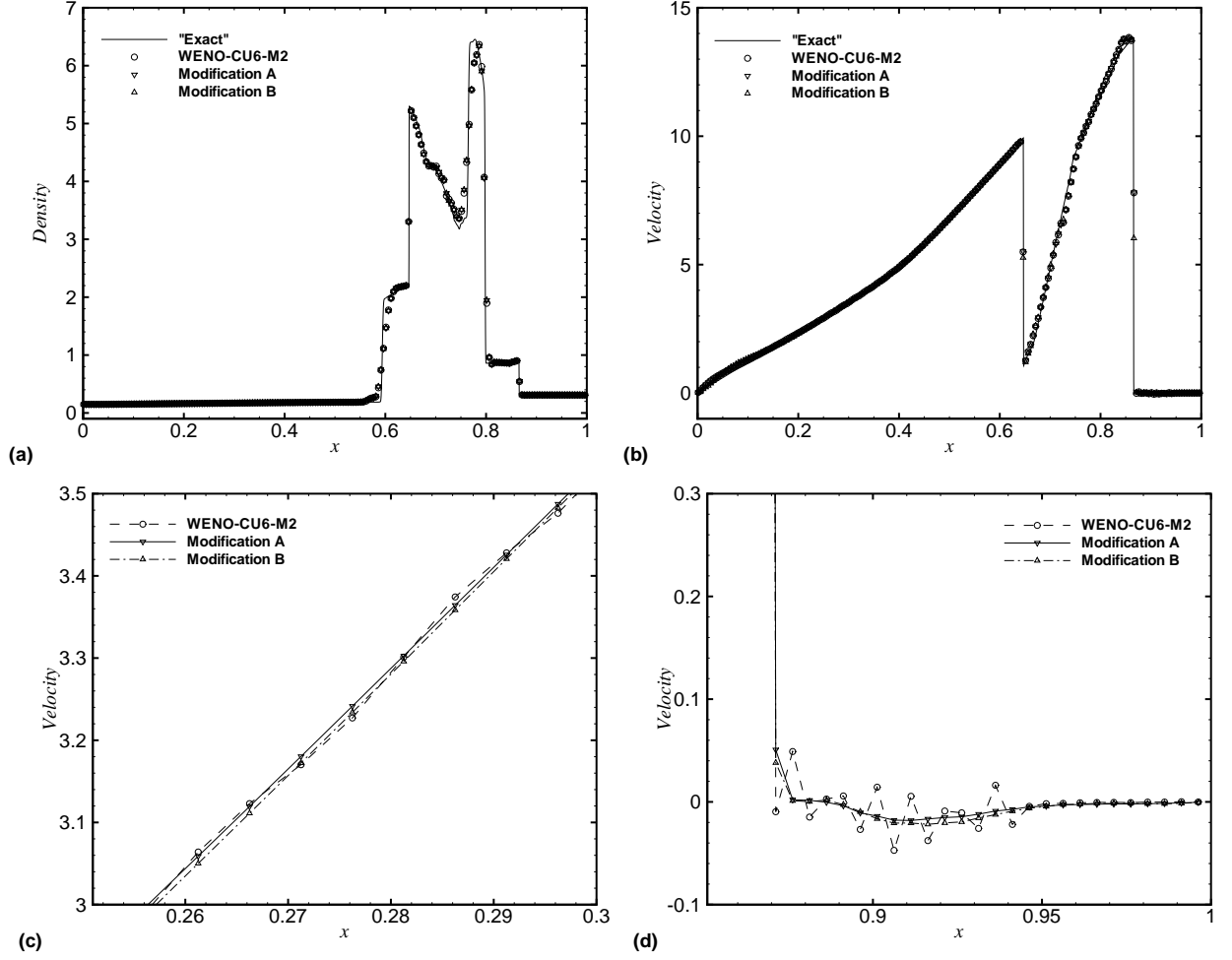


Figure 2. Numerical solution of the two-blast-wave interaction problem at  $t = 0.038$  on a 400 points grid: (a) density and (b) velocity profiles, (c) and (d) close-up views of the velocity profile.